THE CHAMBER STRUCTURES OF THE GROTHENDIECK GROUPS COMING FROM BRICKS

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Abstract. We consider the real-valued Grothendieck group $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ of the category $\mathsf{proj}\,A$ of finite-dimensional projective modules over an algebra A over a field K. Each element of the Grothendieck group determines a semistabilty condition, which was introduced by King. Following Brüstle–Yang–Treffinger, we can associate a subset of the Grothendieck group to each brick by using semistability conditions, and define a chamber structure of the Grothendieck group. In this proceeding, we give our new results on the chamber structure.

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Notation

In this proceeding, K is a field, and A is a finite-dimensional algebra over the field K. We write $\mathsf{proj}\,A$ for the category of finite-dimensional projective A-modules, and $P_1, \ldots, P_n$ for all non-isomorphic indecomposable projective A-modules. Similarly, $\mathsf{mod}\,A$ denotes the category of finite-dimensional A-modules, and $S_1, \ldots, S_n$ denote all non-isomorphic simple A-modules. We also assume that S_i is the top of P_i, that is, there is a surjection $P_i \to S_i$. For an exact or triangulated category $\mathcal{C}$, the Grothendieck group of $\mathcal{C}$ is denoted by $K_0(\mathcal{C})$.

1. Euler form

In this section, we deal with some fundamental facts on Euler form. We first recall the following well-known facts on the Grothendieck groups of $\mathsf{proj}\,A$ and $\mathsf{mod}\,A$.

Proposition 1. *The following assertions hold.*

1. *The family $(P_i)_{i=1}^n$ is a $\mathbb{Z}$-basis of $K_0(\mathsf{proj}\,A)$ and $K_0(\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A))$.*
2. *The family $(S_i)_{i=1}^n$ is a $\mathbb{Z}$-basis of $K_0(\mathsf{mod}\,A)$ and $K_0(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A))$.*

For these two Grothendieck groups, we have a bilinear-form called Euler form.

Definition 2. We define *Euler form* $\langle ?, ? \rangle \colon K_0(\mathsf{proj}\,A) \times K_0(\mathsf{mod}\,A) \to \mathbb{Z}$ by

$$\langle T, X \rangle := \sum_{k \in \mathbb{Z}} (-1)^k \dim_K \mathsf{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)}(T, X[k])$$

for $T \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ and $X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$.

The detailed version of this paper will be submitted for publication elsewhere.

The families $(P_i)_{i=1}^n$ and $(S_i)_{i=1}^n$ give dual bases with respect to Euler form.

Proposition 3. *The families $(P_i)_{i=1}^n$ and $(S_i)_{i=1}^n$ satisfy*

$$\langle P_i, S_j \rangle = \begin{cases} \dim_K \operatorname{End}_A(S_j) & (i = j) \\ 0 & (i \neq j) \end{cases}$$

for $i, j \in \{1, \ldots, n\}$.

We can find other dual bases by using

- (2-term) *silting objects* in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$, and
- (2-term) *simple-minded collections* in $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$.

See [10, 5, 2] for the definitions of these two notions. We write (2-)$\operatorname{silt} A$ for the set of isoclasses of basic (2-term) silting objects in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$, and similarly, (2-)$\operatorname{smc} A$ denotes the set of (2-term) simple-minded collections in $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)$. On these notions, Koenig–Yang [10] and Brüstle–Yang [5] obtained the next results.

Proposition 4. *We have the following bijections.*

(1) [10, Theorem 6.1] *There exists a bijection $\operatorname{silt} A \to \operatorname{smc} A$ sending each silting object T to the set of isoclasses of simple objects in the abelian category*

$$T[\neq 0]^{\perp} := \{X \in \mathsf{D}^{\mathrm{b}}(\operatorname{mod} A) \mid \operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)}(T[k], X)\}.$$

(2) [5, Corollary 4.3] *The bijection in* (1) *is restricted to a bijection $2\text{-}\operatorname{silt} A \to 2\text{-}\operatorname{smc} A$.*

We can use the bijection in (1) to construct dual bases. By [10, Lemma 5.3], we have the following property (see also [2, Theorem 3.17]).

Proposition 5. *Let $T \in \operatorname{silt} A$ correspond to $X \in \operatorname{smc} A$. Then, there exist families $(T_i)_{i=1}^n$ and $(X_i)_{i=1}^n$ satisfying the following conditions:*

- $T = \bigoplus_{i=1}^n T_i$,
- $\mathcal{X} = \{X_i\}_{i=1}^n$, *and*
- $(T_i)_{i=1}^n$ *and* $(X_i)_{i=1}^n$ *give dual bases with respect to Euler form; more precisely,*

$$\langle T_i, X_j \rangle = \begin{cases} \dim_K \operatorname{End}_{\mathsf{D}^{\mathrm{b}}(\operatorname{mod} A)}(X_j) & (i = j) \\ 0 & (i \neq j) \end{cases}.$$

Therefore, for each $T \in 2\text{-}\operatorname{silt} A$ sent to $\mathcal{X} \in 2\text{-}\operatorname{smc} A$, we take families $(T_i)_{i=1}^n$ and $(X_i)_{i=1}^n$ satisfying the three conditions above. We assume this setting in the rest of this proceeding.

2. Cones of silting objects

Now, we consider the real-valued Grothendieck group $K_0(\operatorname{proj} A)_{\mathbb{R}} := K_0(\operatorname{proj} A) \otimes_{\mathbb{Z}} \mathbb{R}$, which is naturally identified with the n-dimensional Euclidean space $\mathbb{R}^n$ by

$$\sum_{i=1}^n g_i[P_i] \mapsto (g_1, g_2, \ldots, g_n).$$

For each object $U = \bigoplus_{i=1}^{m} U_i \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ with U_i indecomposable, we define a cone $C(U)$ in the Euclidean space $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ by

$$C(U) := \left\{ \sum_{i=1}^{m} a_i [U_m] \mid a_1, \ldots, a_m \in \mathbb{R}_{\geq 0} \right\}.$$

We will mainly consider the case that U is a 2-term silting object.

The intersection of the cones of two 2-term silting objects expresses their common direct summands.

Proposition 6. [8, Corollary 6.7] (see also [6, 7]) *Let $T, T' \in \mathsf{2\text{-}silt}\, A$ and $\mathsf{add}\, T \cap \mathsf{add}\, T' = \mathsf{add}\, U$ with $U \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$. Then, $C(T) \cap C(T') = C(U)$.*

We also have other basic properties of the cones as follows.

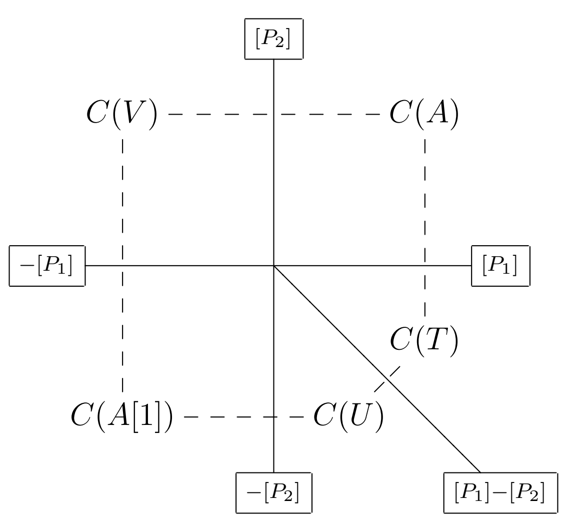
- The cone $C(T)$ has exactly n walls $C(T/T_i)$ with $i \in \{1, \ldots, n\}$.
- Each wall $C(T/T_i)$ is $(n-1)$-dimensional.
- Each wall $C(T/T_i)$ corresponds to the mutation of T at T_i.
- Each wall $C(T/T_i)$ is orthogonal to $[X_i] \in K_0(\mathsf{mod}\, A)$ with respect to Euler form.
- If $T, T' \in \mathsf{2\text{-}silt}\, A$ are non-isomorphic, then $C(T)^\circ \cap C(T') \neq \emptyset$, where $C(T)^\circ$ is the interior of $C(T)$.

Let us give an example.

Example 7. Let A be the path algebra $K(1 \to 2)$. Then, the basic 2-term silting objects in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\, A)$ are the following five objects:

$$\begin{aligned} A &= P_1 \oplus P_2, \\ T &= P_1 \oplus (P_2 \to P_1), \\ U &= P_2[1] \oplus (P_2 \to P_1), \\ V &= P_1[1] \oplus P_2, \\ A[1] &= P_1[1] \oplus P_2[1]. \end{aligned}$$

The cones of these objects are displayed as in the picture below:



To investigate the connection between the 2-term silting objects and their cones, we use the numerical torsion(-free) classes introduced by Baumann–Kamnitzer–Tingley [3]. We regard each $\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ as a $\mathbb{Z}$-linear form $\theta := \langle \theta, ? \rangle \colon K_0(\mathsf{mod}\, A) \to \mathbb{R}$.

Definition 8. [3, Subsection 3.1] For $\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$, we define the *numerical torsion class* $\overline{\mathcal{T}}_\theta$ by

$$\overline{\mathcal{T}}_\theta := \{M \in \mathsf{mod}\, A \mid \text{for any quotient module } N,\, \theta(N) \geq 0\}.$$

Dually, the *numerical torsion-free class* $\overline{\mathcal{F}}_\theta$ is defined by

$$\overline{\mathcal{F}}_\theta := \{M \in \mathsf{mod}\, A \mid \text{for any submodule } L,\, \theta(L) \leq 0\}.$$

The pair $(\overline{\mathcal{T}}_\theta, \overline{\mathcal{F}}_\theta)$ is not necessarily a torsion pair in $\mathsf{mod}\, A$. It is a torsion pair if and only if $\overline{\mathcal{T}}_\theta \cap \overline{\mathcal{F}}_\theta$ is $\{0\}$.

In order to explain the importance of numerical torsion(-free) classes, we recall the following significant fact in τ-tilting theory on functorially finite torsion(-free) classes from [1].

Remark 9. [1, Theorem 3.2] There exist bijections

$$\begin{aligned} \mathsf{2\text{-}silt}\, A &\to \{\text{functorially finite torsion classes}\}, \\ T &\mapsto \mathcal{T}_T := \mathsf{Fac}\, H^0(T); \\ \mathsf{2\text{-}silt}\, A &\to \{\text{functorially finite torsion-free classes}\}, \\ T &\mapsto \mathcal{F}_T := \mathsf{Sub}\, H^{-1}(\nu T). \end{aligned}$$

Moreover, $(\mathcal{T}_T, \mathcal{F}_T)$ is a torsion pair in $\mathsf{mod}\, A$.

Yurikusa [11] showed that any functorially finite torsion(-free) class is realized numerically.

Proposition 10. [11, Theorem 1.3] *Let $T \in \mathsf{2\text{-}silt}\, A$ and $\theta \in C(T)^\circ$. Then, $\overline{\mathcal{T}}_\theta = \mathcal{T}_T$ and $\overline{\mathcal{F}}_\theta = \mathcal{F}_T$.*

In particular, the numerical torsion(-free) class is constant in the interior $C(T)^\circ$ of the cone $C(T)$. Inspired by this property, we introduce an equivalence for elements in the real-valued Grothendieck group.

Definition 11. Let $\theta, \theta' \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$. Then, we say that θ and θ' are *TF equivalent* if $\overline{\mathcal{T}}_\theta = \overline{\mathcal{T}}_{\theta'}$ and $\overline{\mathcal{F}}_\theta = \overline{\mathcal{F}}_{\theta'}$. In this case, we write $\theta \sim \theta'$. The TF equivalent class which θ belongs is denoted by $[\theta]$.

By using the result by Yurikusa above and some of our results in [2], we can show the following property.

Proposition 12. *For each $T \in \mathsf{2\text{-}silt}\, A$, the interior $C(T)^\circ$ is a TF equivalent class.*

3. Semistable subcategories and the walls for modules

In general, the cones $C(T)$ do not cover the real-valued Grothendieck group $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$, so we shall extend the observation on TF equivalent classes outside the cones in this section. For this purpose, we use the semistability of modules introduced by King.

Definition 13. [9, Definition 1.1] For $\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$, we define the *$\theta$-semistable subcategory* $\mathcal{W}_\theta \subset \mathsf{mod}\, A$ by

$$\mathcal{W}_\theta := \mathcal{T}_\theta \cap \mathcal{F}_\theta \subset \mathsf{Ker}\langle \theta, ? \rangle.$$

We remark some properties which are easily deduced.

Remark 14. For $\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$, we have the following assertions.

(1) Let $\theta, \theta' \in K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ be TF equivalent. Then, $\mathcal{W}_{\theta''}$ is constant for the points θ'' in the line segment $[\theta, \theta']$.
(2) (deduced from Proposition 10) Let $T \in \mathsf{2\text{-}silt}\, A$ and $\theta \in C(T)^\circ$. Then, $\mathcal{W}_\theta = \{0\}$.

The subcategory $\mathcal{W}_\theta$ is a wide subcategory of $\mathsf{mod}\, A$. Thus, each simple object S of $\mathcal{W}_\theta$ is a brick, that is, $\mathsf{End}_A(S)$ is a division K-algebra. To display the semistable subcategories on the Euclidean space, we associate a wall to each brick.

Definition 15. [4, Definition 3.2] For each brick S, we define the wall $\Theta_S \subset K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ associated to S by

$$\Theta_S := \{\theta \in K_0(\mathsf{proj}\, A)_{\mathbb{R}} \mid S \in \mathcal{W}_\theta\} \subset \mathsf{Ker}\langle ?, S \rangle.$$

We can consider a chamber structure on the real-valued Grothendieck group $K_0(\mathsf{proj}\, A)_{\mathbb{R}}$ defined by the walls Θ_S for bricks S. To observe the connection between the walls Θ_S and the walls of the cones $C(T)$, we first remark the following facts.

Remark 16. The following assertions hold.

(1) (deduced from Remark 14) For $T \in \mathsf{2\text{-}silt}\, A$ and any brick S, the intersection $C(T)^\circ \cap \Theta_S = \emptyset$ is empty.
(2) [5, Remark 4.11] Any element X in $\mathcal{X} \in \mathsf{2\text{-}smc}\, A$ has a brick S such that $X = S$ or $X = S[1]$.

Remember that we have taken families $(T_i)_{i=1}^n$ and $(X_i)_{i=1}^n$ satisfying the three conditions in Proposition 5 for each $T \in \mathsf{2\text{-}silt}\, A$ sent to $X \in \mathsf{2\text{-}smc}\, A$.

Proposition 17. *In the setting above, let $i \in \{1, \ldots, n\}$ and take a brick S so that $X_i = S$ or $X_i = S[1]$. Then, the wall $C(T/T_i)$ of the cone $C(T)$ is contained in the wall Θ_S associated to the brick S.*

We remark that the wall $C(T/T_i)$ does not coincide with Θ_S in general.

Example 18. In example 7, the bricks in $\mathsf{mod}\, A$ are S_2, P_1, S_1. The walls associated to them are

$$\Theta_{S_2} = \mathbb{R}[P_1], \quad \Theta_{P_1} = \mathbb{R}_{\geq 0}([P_1] - [P_2]), \quad \Theta_{S_1} = \mathbb{R}[P_2].$$

For example, the wall $C(A/P_1) = C(P_2) = \mathbb{R}_{\geq 0}[P_2]$ of the cone $C(A)$ is contained in Θ_{S_1}, but they are not equal.

4. Main result

In this section, we give our new results on the chamber strucuture given by the walls associated to bricks.

Recall that any $T \in \mathsf{2\text{-}silt}\, A$ gives a TF equivalent class $C(T)^\circ$, which is an open set in $K_0(\mathsf{proj}\, A)_\mathbb{R}$. Actually, any TF equivalent class whose interior is nonempty can be obtained in this way.

Proposition 19. *Let $\theta \in K_0(\mathsf{proj}\, A)_\mathbb{R}$, then the following conditions are equivalent.*

(a) *The TF equivalent class $[\theta]$ is an open set in $K_0(\mathsf{proj}\, A)_\mathbb{R}$.*
(b) *The interior $[\theta]^\circ$ of the TF equivalent class $[\theta]$ is nonempty.*
(c) *There exists $T \in \mathsf{2\text{-}silt}\, A$ such that $\theta \in C(T)^\circ$.*

As a consequence, we have the following bijection.

Corollary 20. *There exists a bijection from $\mathsf{2\text{-}silt}\, A$ to the set of TF equivalent classes with nonempty interiors sending T to $C(T)^\circ$.*

On the other hand, we also get a criterion of TF equivalence.

Proposition 21. *Let $\theta \neq \theta'$ be distinct elements in $K_0(\mathsf{proj}\, A)_\mathbb{R}$, then the following conditions are equivalent.*

(a) *The elements θ and θ' are TF equivalent.*
(b) *The θ''-semistable subcategory $\mathcal{W}_{\theta''}$ is constant for $\theta'' \in [\theta, \theta']$.*
(c) *There does not exist a brick S such that $\Theta_S \cap [\theta, \theta']$ consists of one point.*

Finally, we can state the main result of this proceeding, which says that there exist no cones $C(T)$ for $T \in \mathsf{2\text{-}silt}\, A$ where the walls Θ_S for bricks S lie densely.

Theorem 22. *As subsets of $K_0(\mathsf{proj}\, A)_\mathbb{R}$,*

$$\coprod_{T \in \mathsf{2\text{-}silt}\, A} C(T)^\circ = K_0(\mathsf{proj}\, A)_\mathbb{R} \setminus \overline{\left(\bigcup_{S\colon \text{ brick}} \Theta_S \right)}.$$

Moreover, the left-hand side is the decomposition to the connected components.

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